Evidences that the Riemann Hypothesis is true

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Abstract

In this paper two evidences are added to the existing in the literature that the Riemann Hypothesis (RH) is true.
The first one shown that one approximated equation to the functional equation of the Riemann Zeta Function has probably all its non trivial zeros in the critical line $\Re(s)=1/2$.
A second evidence and more important, is obtained from expanding in a Taylor Series the functional equation of the Riemann Zeta function expressed as a Fourier Transform. This evidence, as the word means, is not a rigorous mathematical proof but perhaps could be a way to prove RH .

1. Introduction

An introduction to the Zeta Riemann Function and to the Riemann Hypothesis can be found in the references [1]-[6] and [9].

The Riemann Zeta Function for $\Re(s) > 1$ is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1a)$$

Riemann proved that $\zeta(s)$ has an analytic continuation to the whole complex plane apart from a simple pole at $s = 1$.

$\zeta(s)$ has an analytic continuation in the critical region $0 < \Re(s) < 1$ and is defined as :

$$\varsigma(s) = \frac{\eta(s)}{1-2^{-s}} \qquad (1b)$$

where $\eta(s)$ is the Eta Function of Dirichlet given by:

$$\eta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^s} \qquad (1c)$$

Riemann proved also that this Zeta function verifies also an amazing functional equation, which in its symmetric form is given by

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s) \qquad (1d)$$

where $\Gamma(s)$ is the Gamma-function.

The above functional equation states that if a complex number s is a zero of $\xi(s)$ then it is also a zero the complex number $1-s$.

The Riemann Zeta function is connected with the prime numbers which can be considered as the "atoms" of integer numbers because the fundamental theorem of arithmetic states that every integer can be factored into primes in a unique way.

This connection, valid for $\Re(s) > 1$, is given by the equation

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (1e)$$

where the infinite product (called the *Euler Product*) is over all prime numbers.

The Riemann Hypothesis states that all non trivial zeros (trivial zeros are the negative even integers: -2,-4.-6….) of $\xi(s)$ satisfy $\Re(s) = 1/2$.

That is, the Riemann Hypothesis establish that all the non trivial (complex) zeros of the Riemann Zeta Function lie in the critical line $\Re[s]=1/2$; then if s is a complex number that verifies $s = \sigma + i\, t$ then all the zeros of the Riemann Zeta Function correspond to points of the complex line with $\sigma=1/2$ for the different values of t.

Riemann proved that all possible non trivial zeros of the Riemann Zeta Function lie in the critical region $0 < \sigma < 1$.

Most of mathematicians believe that the Riemann Hypothesis is true, but although exist numerous attempts to prove it since his formulation [10] any proof has been admitted for the scientific community.

As pointed out by Brian Conrey [9] there are some evidence to believe that the Riemann Hypothesis is true:

a) Billions of zeros verify the Riemann Hypothesis. Work of van de Lune, Sebastian Wedeniswski and Andrew Odlyzko have proved that the first 100 billions of zeros are on the critical line and also millions of zeros near numbers 10^{20}, 10^{21} and 10^{22} agree with the Riemann Hypothesis.

b) Almost all of the zeros are very near the 1/2 –line. In fact, it has been proved that more than 99 percent of zeros $s = \sigma + it$ satisfy $|\sigma - 1/2| \leq 8/\log|t|$

c) It has been proved that a proportion of zeros are on the critical line. Selberg got a positive proportion, and N.Levison [7] showed at least1/3; that proportion has been improved by B.Conrey [8] to 40 percent. Also Riemann Hypothesis (RH) implies that all zeros of all derivatives of $\xi(s)$ are on the 1/2 –line.It has been shown that more that 99 percent of the zeros of the third derivative $\xi'''(s)$ are on the 1/2-line.

d) Although the primes are distributed apparently in a random way they hide a symmetry related to the Riemann Zeta function and expressed by the Euler product of equation (1e). If RH were false, there would be some strange irregularities in the distribution of primes; the first zero off the line would be a very important constant, and that seems unlike.

In this paper is presented two more arguments to believe that the RH is true. They are based on the Fourier transform of the functional equation of the Riemann Zeta function

2. First Evidence: An approximated Fourier Transform function to the functional equation of Riemann Zeta Function with only real zeros

Let $\Xi(t) = \xi(1/2 + it)$. It is know (see Titchmarsh [5] chapter 10) that

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x) e^{itx} dx \qquad (2a)$$

where

$$\Phi(x) = \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9x/2} - 3n^2\pi e^{5x/2}\right) \exp\left(-n^2\pi e^{2x}\right) \qquad (2b)$$

$\Phi(x)$ is a even function and $\Xi(t)$ is also an even function and is real for real t.

The RH is true if and only if the Fourier transform $\Xi(t)$ has only real zeros.

Pólya [11] and D.Cardon [12] have studied the reality of zeros of various Fourier transforms. One idea to prove the RH is systematically study classes of reasonable functions whose Fourier transforms have all real zeros and then try to prove that $\Xi(t)$ is in the class.

A first approximation of $\Phi(x)$ is given by [5]

$$\Phi_0(x) = \left(2\pi \cosh(9x/2) - 3\cosh(5x/2)\right) \exp\left(-2\pi \cosh(2x)\right) \qquad (2c)$$

The Fourier transform (equation (2a)) of $\Phi_0(x)$ has all zeros real .

Further de Bruijn, Newman, D.Hejhal and others have investigated the ideas of Pólya. Hejhal (1990) has shown that almost all of the zeros of the Fourier transform of any partial sum of $\Phi(x)$ are real, so this finding is also an evidence of the truth of the RH.

This paper are related with the above investigations and begins defining the function $\Phi_1(x)$, constructed looking at the reference function $\Phi_0(x)$, and given by

$$\Phi_1(x) = \beta \cosh(\alpha_1 x) \exp(-2\pi \cosh(\alpha_2 2x)) \qquad \text{(2d)}$$

where β, α_1 and α_2 are parameters obtained from a fit of the function $\Phi_1(x)$ to the function $\Phi(x)$ given by equation (2b). The fit is performed using a non linear regression with the Levenverg- Marquardt algorithm and is shown in Figure 1. The results for the parameters obtained from the fit are in the following Table I.

	$\Phi_1(x)$
α_1	2.3025 $\pm$ 0.0004
α_2	1.9545 $\pm$ 0.00006
β	239.1811 $\pm$ 0.0013

Table I

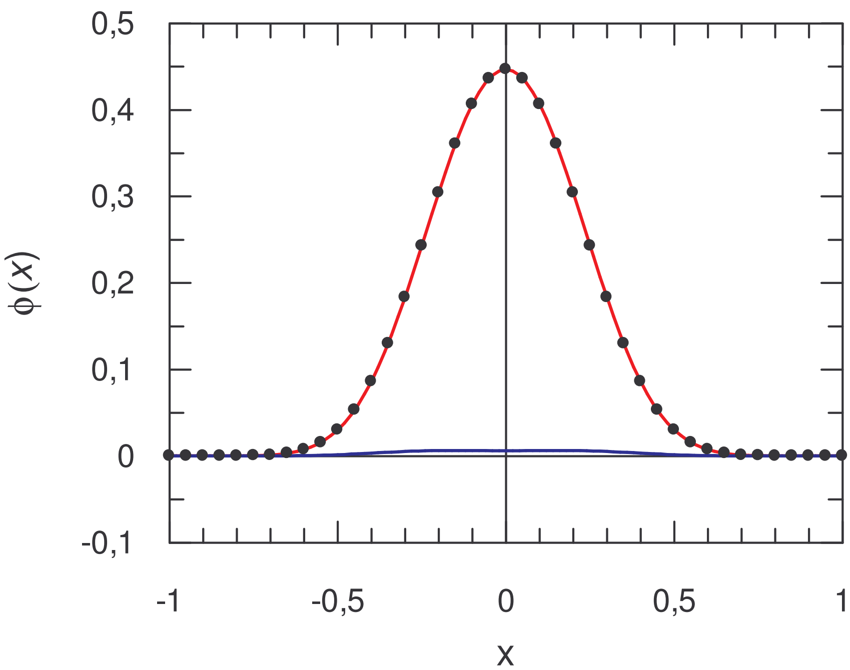

Figure 1. *The filled circles represent $\Phi(x)$ given by equation (2b) calculated using 100 terms in the series. The blue line represents $\Phi_0(x)$ given by the equation (2c). $\Phi_1(x)$ given by the equation (2d), and using the parameters of Table I, is represented by the red line. The non linear fit was obtained from 1600 points equally spaced between x=-1 and x=1 of $\Phi(x)$ calculated using 100 terms in the series. The reduced chi square of the fit was $9.26.10^{-10}$ for $\Phi_1(x)$.*

As can be seen in Figure 1 the fit are very good and the functions $\Phi_1(x)$ presents a high approximation to the function $\Phi(x)$ for lower values of x. However, as can be seen in Figure 2 , at greater absolute values of x, $\Phi_0(x)$ is more approximate to $\Phi(x)$ than $\Phi_1(x)$ to $\Phi(x)$.

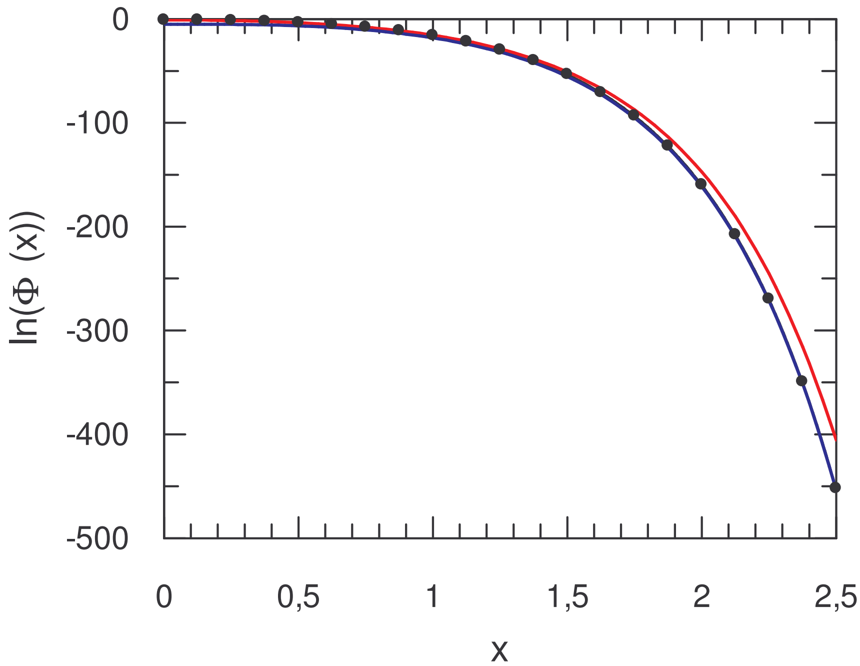

Figure 2. *The filled circles correspond to* $\ln(\Phi(x))$, *the blue line* to $\ln(\Phi_0(x))$ *and the red line to* $\ln(\Phi_1(x))$.

In Figure 2 were not represented negative values of x because the computer calculations give, for $x< -1.2$, results that indicate $\Phi(x)$ in not an even function(are obtained also negative random values). The cause is probably the lower precision of the computer calculations (double precision is not enough) (If the calculations were correct the functional equation of the Riemann Zeta Function will be erroneous)

In the Figure 3 are represented $\Xi(t)$, $\Xi_0(t)$ and $\Xi_1(t)$. This last two function correspond to change $\Phi(x)$ in equation (2a) by $\Phi_0(x)$ and $\Phi_1(x)$, respectively.

Since $\Phi(x)$, $\Phi_0(x)$ and $\Phi_1(x)$ are even functions the equation (2a), for real t, can be also expressed as

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\cos(tx)dx \qquad (2e)$$

The integral was calculated using the Simpson Method of integral calculation with the appropriate values the integration interval. The calculations were performed for values of t from 0 to 37.5 so can be observed the first few zeros of the Riemann Zeta Function.

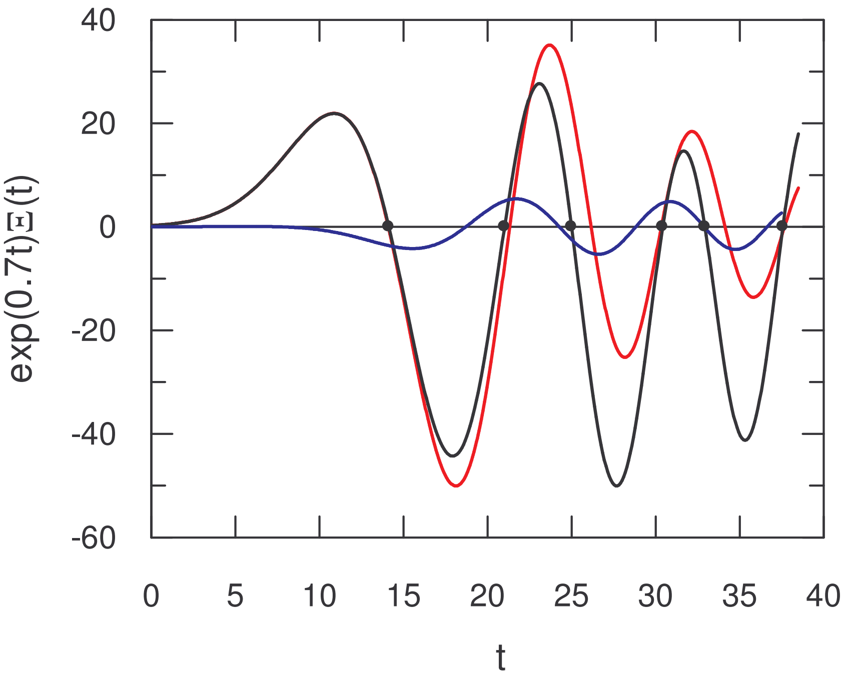

Figure 3. *The black line correspond to* Ξ(t)*; the blue line to* Ξ_0(t) *and the red line to* Ξ_1(t). *The filled circles represent the Zeros of the Riemann Zeta function found in the literature. The integrals were calculated using the Simpson Method choosing 4000 points of integration in the interval (-2,2).(As the values of t increases it is necessary choose more points of integration to obtain precision in the results) The graphic is scaled with the function* exp(0.7t) *since* Ξ(t) *decreases exponentially with t.*

From the figure 3 one can deduce that in the calculation of the integral of equation (2e) the greater values of *x* are important when *t* increases. For lower values of t only the values of $\Phi(x)$ until about $x=\pm 1$ are important. This can be corroborated in Figures 1 and 3 because in the interval (-1,1) $\Phi(x)$ and $\Phi_1(x)$ are approximate equals and Ξ(t) and Ξ_1(t) are practically coincident for lower values of t (less than 15) and the difference between them increases with *t*. As can be seen also in the Figure 3, Ξ_0(t) is less approximate than Ξ_1(t) to Ξ(t) for lower values of *t*. One can also predict, take into account the results of Figure 2, that if *t* increases then Ξ_0(t) is more approximated to Ξ(t).

To deduce if the zeros of Ξ_1(t) are real one can use one theorem due to de Bruijn [9].

Theorem (de Bruijn): *Let f(x) be an even non constant entire function of x such that f(x) ≥ 0 for real x and* $f'(x) = \exp(\gamma x^2) g(x)$, *where* $\gamma \geq 0$ *and g(x) is an entire function of genus ≤ 1 with purely imaginary zeros only. Then* $\Psi(t) = \int_{-\infty}^{\infty} \exp(-f(x)) e^{itx} dx$ *has real zeros only.*

Using the above theorem for $\Xi_1(t)$ then $exp(-f(x))=\Phi_1(x)$ (Since $\Phi_1(x)$ is an even non constant function then $f(x)$ is also an even non constant function. Moreover $0 < \Phi_1(x) <1$ for real x, so $f(x) \geq 0$ for real x. This implies that $f(x)$ verifies the conditions of the de Bruijn Theorem). If one considers also that $\gamma = 0$ then

$$g(x) = -\frac{\Phi_1'(x)}{\Phi_1(x)} \qquad (2f)$$

From equation (2d) and (2f) one can deduce that

$$g(x) = -\{\alpha_1 \tanh(\alpha_1 x) - 2\pi \sinh(\alpha_2 x)\} \qquad (2g)$$

To prove that $g(x)$ has only purely imaginary zeros as requires the above theorem then if $x = a + ib$ one arrives that is necessary that the two following equations will be satisfied only when $a=0$

$$g_r(x = a+ib) = \alpha_1 \sinh(\alpha_1 a)\cos(\alpha_1 b) - 2\pi\alpha_2\{\cosh(\alpha_1 a)\cos(\alpha_1 b)\sinh(\alpha_2 a)\cos(\alpha_2 b) - \sinh(\alpha_1 a)\sin(\alpha_1 b)\cosh(\alpha_2 a)\sin(\alpha_2 b)\} = 0 \qquad (2h\ 1)$$

$$g_i(x = a+ib) = \alpha_1 \cosh(\alpha_1 a)\sin(\alpha_1 b) - 2\pi\alpha_2\{\sinh(\alpha_1 a)\sin(\alpha_1 b)\sinh(\alpha_2 a)\cos(\alpha_2 b) + \cosh(\alpha_1 a)\cos(\alpha_1 b)\cosh(\alpha_2 a)\sin(\alpha_2 b)\} = 0 \qquad (2h\ 2)$$

Since, when $a = 0$, $g_r(b)=0$ for all b and accordingly the zeros of $g_i(x)$ verify the equation

$$g_i(b) = \alpha_1 \sin(\alpha_1 b) - 2\pi\alpha_2 \cos(\alpha_1 b)\sin(\alpha_2 b) = 0 \qquad (2h\ 3)$$

The zeros of $g_r(x)$ and $g_i(x)$ do not depend on the value of a so is only necessary prove for a particular value of a that the zeros of $g(x)$ are only purely imaginary zeros.. In the Figure 4, which is the graph of the above two functions for $a=0.1$, is observed for positive values of b (It is not necessary represent the negative values because the graphic is symmetric) that the zeros of $g_r(x)$ and $g_i(x)$ are not the same in the scale represented in the figure. It seems that for all values of b the zeros are purely imaginary zeros, verifying then the Theorem of de Bruijn if $g(x)$, given by equation (2g), is an entire function of genus ≤ 1.

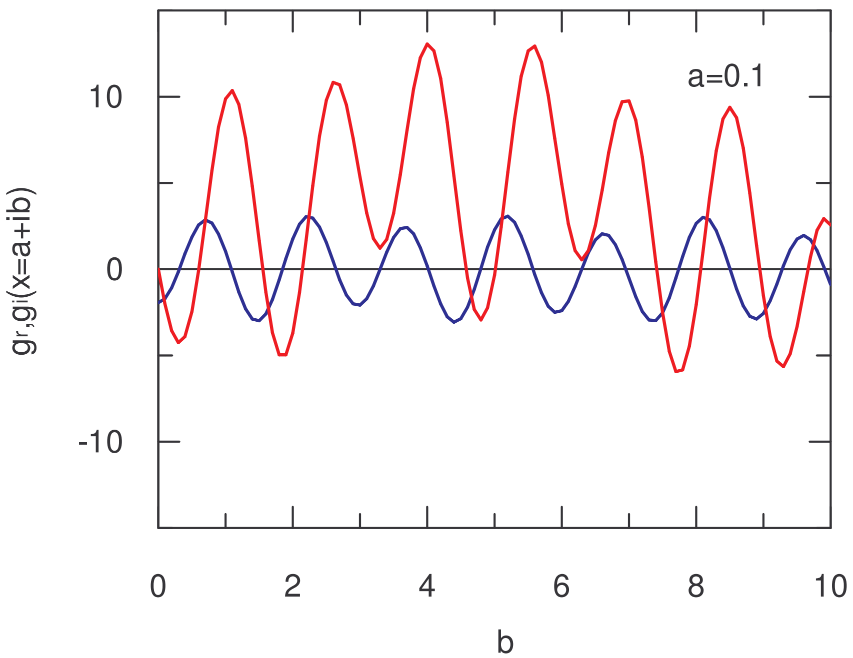

Figure 4. *The red line represents $g_r(x)$ for a=0.1 and the blue line correspond to $g_i(x)$ also for a=0.1, given by equations (2h 1) and (2h 2), respectively.*

In summary, the function $\Xi_0(t)$ has zeros only for real values of t and probably occurs the same for $\Xi_1(t)$. Both functions verify the symmetric equation which satisfy the functional equation of the Riemann Zeta function, that is , $\Xi(s)= \Xi(1-s)$, $\Xi_0(s)= \Xi_0(1-s)$ and $\Xi_1(s)= \Xi_1(1-s)$. The zeros of the three functions are quasi random and have similar values of t. If the zeros of the two approximated functions (one function that is approximated for lower values of t and the other for higher values of t) are real then why should not be real the zeros of $\Xi(t)$? It seems that there is no reason for exist complex zeros of $\Xi(t)$. If this is not the case then the Riemann Zeta function will be a very special function and the reason could be the relation between the Riemann Zeta function and the prime numbers expressed in the Euler product of equation (1e) which relation do not present the other two functions.

3. Second Evidence. The functional equation of the Riemann Zeta Function expressed as a Fourier Transform and expanded in a Taylor Series.

If one considers that $s=1/2+it=\sigma + iT$ then t in the equation $\Xi(t)=\xi(1/2+it)$ is given by $t=T-i(\sigma-1/2)$ and the equation (2a) is transformed into

$$\Xi(s)=\xi(s)=\int_{-\infty}^{\infty}\Phi(x)e^{(\sigma-1/2)x}e^{iTx}dx \qquad (3a)$$

Now if $p=\sigma-1/2+iT$ the above equation read as

$$\Xi(p) = \xi(p) = \int_{-\infty}^{\infty} \Phi(x) e^{px} dx \quad (3b)$$

The integrand of the equation (3b) can be transformed in a even function resulting

$$\Xi(p) = \xi(p) = \frac{1}{2} \int_{-\infty}^{\infty} \Phi(x) \left(e^{px} + e^{-px} \right) dx \quad (3c)$$

The above integral can be simplified into

$$\Xi(p) = \xi(p) = \int_{-\infty}^{0} \Phi(x) \left(e^{px} + e^{-px} \right) dx \quad (3d)$$

On the other hand $\Phi(x)$ of equation (2b) can be expressed also as:

$$\Phi(x) = \sum_{n=1}^{\infty} \left(a_n e^{9x/2} - b_n e^{5x/2} \right) \exp\left(-c_n e^{2x} \right) \quad (3e)$$

where

$$a_n = 2n^4 \pi^2 \quad (3e\ 1)$$

$$b_n = 3n^2 \pi \quad (3e\ 2)$$

$$c_n = n^2 \pi \quad (3e\ 3)$$

If $\exp\left(-c_n e^{2x}\right)$ is expanded in a Series of Taylor then the equation (3e) results

$$\Phi(x) = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (c_n)^j \left\{ a_n e^{(9/2+2j)x} - b_n e^{(5/2+2j)x} \right\} \quad (3f)$$

Using the equation (3f) into the integral of equation (3d) then results

$$\Xi(p) = \xi(p) = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (c_n)^j \left\{ a_n \left(\frac{1}{9/2+2j+p} + \frac{1}{9/2+2j-p} \right) - b_n \left(\frac{1}{5/2+2j+p} + \frac{1}{5/2+2j-p} \right) \right\} \quad (3g)$$

The equation (3g), if

$$\Xi(p) = \Xi(s) = \Xi^r(s) + i\Xi^i(s) \quad (3h)$$

leads to

$$\Xi^r(s) = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(c_n)^j\left\{a_n(9+4j)\left(\frac{h_j^{\,1}(s)}{\left(h_j^{\,1}(s)\right)^2+\left(w(s)\right)^2}\right)-\right.$$

$$\left. b_n(5+4j)\left(\frac{h_j^{\,2}(s)}{\left(h_j^{\,2}(s)\right)^2+\left(w(s)\right)^2}\right)\right\} \qquad (3h1)$$

$$\Xi^i(s) = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(c_n)^j\left\{a_n(9+4j)\left(\frac{1}{\left(h_j^{\,1}(s)\right)^2+\left(w(s)\right)^2}\right)-\right.$$

$$\left. b_n(5+4j)\left(\frac{1}{\left(h_j^{\,2}(s)\right)^2+\left(w(s)\right)^2}\right)\right\}w(s) \qquad (3h2)$$

where

$$h_j^{\,1}(s) = (9/2+2j)^2 + T^2 - (\sigma - 1/2)^2 \qquad (3h3)$$

$$h_j^{\,2}(s) = (5/2+2j)^2 + T^2 - (\sigma - 1/2)^2 \qquad (3h4)$$

$$w(s) = 2(\sigma - 1/2)T \qquad (3h5)$$

The Riemann Hypothesis is true if $\Xi(s)$ is zero only when $\sigma = 1/2$.
When $\sigma = 1/2$ $\Xi(s)$ is real since $w(s)=0$ and accordingly $\Xi^i(s)=0$.

One can observe from equations (3h) the terms of the series Ξ^r(s) and Ξ^i(s)/$w(s)$ are identical except in that the first one contains the terms h_j^1 and h_j^2 in the numerator of the fractions of these equations in place of 1 for the second one. Then it seems unlikely that for a given value of T and $\sigma \neq 1/2$ both series converges to the same value. This would imply that will not exist any value of T and $\sigma \neq 1/2$ for which Ξ^r(s) and Ξ^i(s) are both zero, so this could be an evidence that the RH is true.

Another evidence, more justified, that the RH is true results from consider that as T increases then the term $(\sigma - 1/2)^2$ in equations (3h3) and (3h4) is less important with respect to T^2 and also $w(s)$ is less important than hj^1 and hj^2, so at very high values of T Ξ^r(s) and Ξ^i(s)/$w(s)$ are practically independent of σ. This implies that the zeros Ξ^r(s) and Ξ^i(s) are also practically independent of σ at high values of T and accordingly if the RH were not true, that is, if would exist a zero of Ξ(s) for a given value of T and $\sigma \neq 1/2$, then will exist infinite approximated zeros for that value of T and for all $\sigma \in$

(0,1). Since it is known almost all of the zeros are very near the 1/2 –line (more than 99 percent of zeros satisfy $|\sigma - 1/2| \le 8/\log|T|$) this implies also one evidence that the RH is true.
The above evidences let the possibility that the RH is false, that is, as the words means, is probably that RH is true but they are no proofs.

However the last argument perhaps could be considered, not as a rigorous mathematical proof, but as an empirical proof when applied to values of $\sigma > 1$, with the condition that $\sigma << T$, because it is known that the Riemann Zeta function and its functional equation have no zeros when $\Re(s)>1$. According to the property of that $\Xi^r(s)$ and $\Xi^i(s)/w(s)$ are practically independent of σ so if would exist a zero of $\Xi(s)$ for a given value of T and $\sigma \neq 1/2$ (in the critical strip $\sigma \in (0,1)$), then will exist infinite approximated zeros for that value of T and for all $\sigma > 1$ (with the condition that $\sigma << T$) and it seems that this is not possible. The condition of $\sigma << T$ for this argument to be valid is justified because it is known that at least for $T < 10^{11}$ the RH is true.

$\Xi^r(s)$ and $\Xi^i(s)$ were calculated using equation (3a) instead of make use of equation (3h) because the convergence is slow in the series (for a moderate number of terms of the two summands in the series this series did not converge).The equation (3a) implies that

$$\Xi^r(s) = \int_{-\infty}^{\infty} \Phi(x) e^{(\sigma - 1/2)x} \cos(Tx)\,dx \qquad (3i1)$$

$$\Xi^i(s) = \int_{-\infty}^{\infty} \Phi(x) e^{(\sigma - 1/2)x} \sin(Tx)\,dx \qquad (3i2)$$

The results are shown in Figures 5 for different values of σ and *T*.

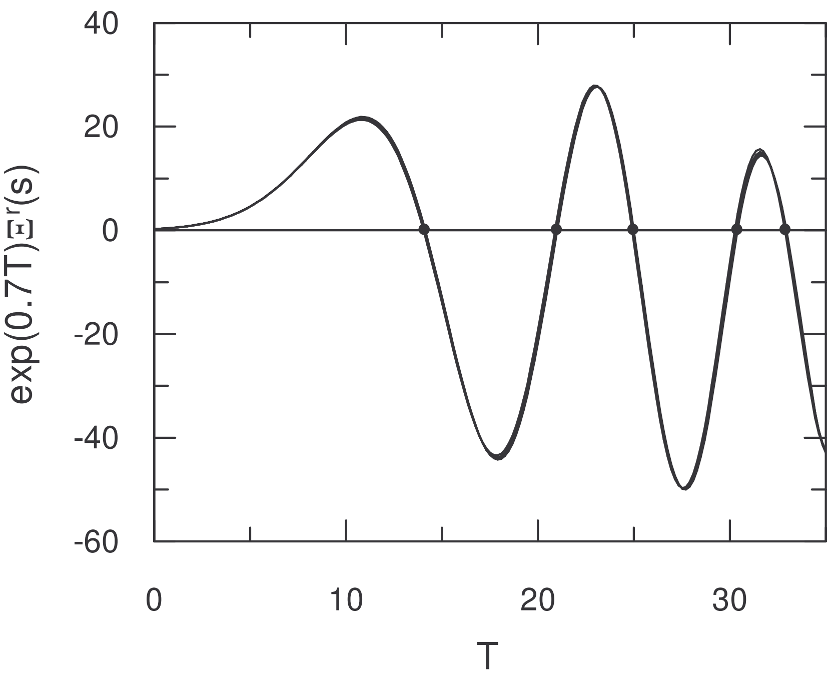

Figure 5a. *$\Xi^r(s)$ as function of T for σ = 0.6, 0.7, 0.8 and 0.9.*

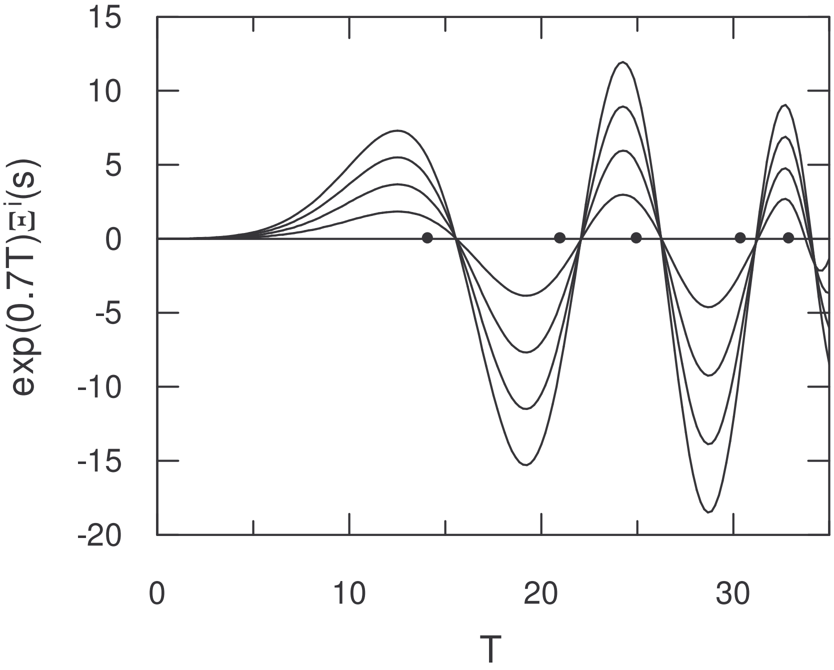

Figure 5b. *$\Xi^{i}(s)$ as function of T for σ = 0.6, 0.7, 0.8 and 0.9.*

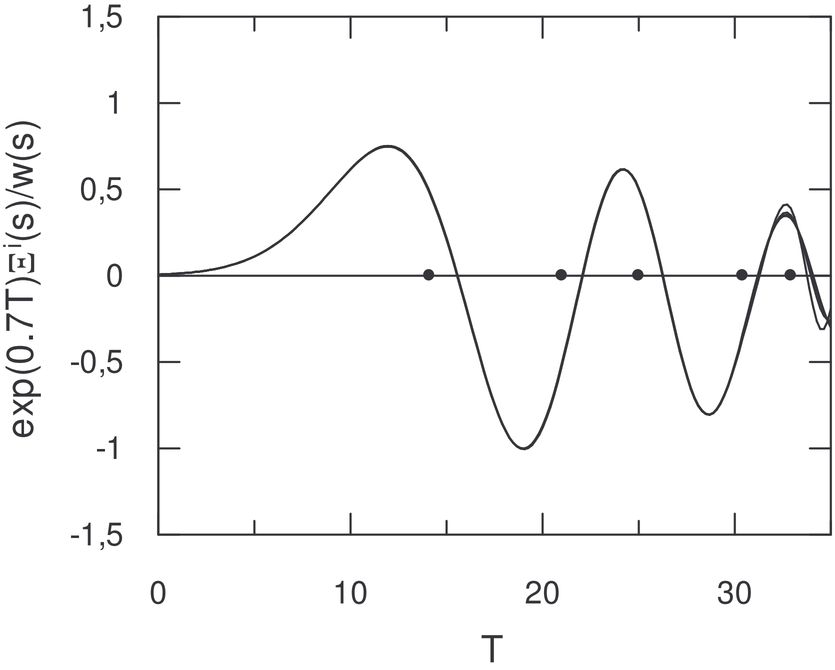

Figure 5c. *$\Xi^{i}(s)/w(s)$ as function of T for σ = 0.6, 0.7, 0.8 and 0.9.*

Figures 5. *$\Xi(s)$ as function of T for the values of σ = 0.6, 0.7, 0.8 and 0.9. The filled circles represent the Zeros of the Riemann Zeta function found in the literature. The integrals were calculated using the Simpson Method choosing 4000 points of integration in the interval (-2,2).(As the values of T increases it is necessary to choose more points of integration to obtain precision in the results, so the results are not very accurate for the higher values of T) The graphic is scaled with the function* $\exp(0.7T)$ *since $\Xi(T)$ decreases exponentially with T.*

The mentioned predictions of equations (3h) are corroborated in the results of the Figures 5. $\Xi^r(s)$ and $\Xi^i(s)/w(s)$ are practically independent of σ (The graphics for different values of σ are nearly superposed) so the zeros are also almost independent of σ. These calculations seem to be correct also because the zeros of $\Xi^r(s)$ are approximately equal to the zeros of the Riemann Zeta function found in the literature. Also the zeros $\Xi^i(s)$, for $\sigma \neq 1/2$, are different from the zeros of $\Xi^r(s)$. Then the reasoning of the second evidence that the RH is true seems to be justified and this could be not a simple evidence but perhaps a way to prove the Riemann Hypothesis.

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